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Scaling theory of self-avoiding crumpled membranes in solution

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We study the scaling theory of the thermodynamics of membrane solutions at finite concentrations. We use the monodisperse continuous Gaussian membrane model with an n-body excluded volume interaction. Without using renormalization group theory, the radius of gyration and osmotic pressure, etc. at finite concentration can be calculated in lower dimensions. [S1063-651X(97)12009-8]

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I. INTRODUCTION

The statistical thermodynamics of fluctuating surfaces has attracted much attention in recent years with applications in many areas of physics such as supermembrane theory [1] in high energy physics, interface in soft condensed matter physics, and membrane problem in biophysics [2]. Since a membrane is a generalization of polymer, many properties of a membrane can be obtained by generalizing polymer theory. One of these studies is the scaling relation. For example, Flory-Huggins theory predicts a $c^2\nu$ chain contribution to the solution energy where c is the concentration and ν the interaction strength. The ''scaling'' theory of de Gennes and co-workers [3] shows a fractional power law, such as $c^{9/4}$, of the interaction energy. In a ''good'' solvent, the mean square end-to-end vector distance displays $\langle R^2\rangle \approx L^{2v}$ with $v=3/5$ theoretically. A good solvent means that the polymer has strong attractive energy with the solvent and is dissolved over a wide range of temperatures. The excluded volume interaction is responsible for the swelling behavior of the overall chain dimensions in a good solvent.

Quite recently, Nelson and co-workers [4,5] have introduced a two-dimensional (2D) continuum model in d-D space to investigate the flexible surface of fixed connectivity in a solution, in a good solvent. Here d is the dimension of external space and D is the dimension of internal space. With this model, a critical line $d^*(D)=4D/(2-D)$ appears in the (d,D) plane. The upper critical dimension d^* is finite only when $0<D<2$. The self-avoidance becomes negligible in d spatial dimensions whenever $d>d^*$. The most important scaling property is the radius of the gyration exponent v, which is defined by $R_G \approx L^v$. When the self-avoidance is negligible, the resulting ''ideal'' exponent is $v=(2-D)/2$. For a stretched manifold, i.e., the manifold overfills the space and self-avoidance stretches the manifold; v is equal to 1. In the self-avoiding regime, v is equal to $(2-D)/2+O(\varepsilon)$, where $\varepsilon=4D-(2-D)d$. However, the direct renormalization group method (RG) is valid only for $D=1$ (i.e., a polymer) and low concentration solution. When $D\neq 1$, a multilocal operator product expansion method [6,7] has recently been developed to show the validity of the RG method and of scaling laws.

Motivated by the general scaling theory of a polymer solution [8], to go beyond the low concentration regime of validity of the RG method, we are thus led to search for a general scaling relation [9] of the scaling theory of a self-avoiding tethered surface at finite concentration in solution, in a good solvent. For example, the low concentration second virial coefficient is obtained in Ref. [5] by using the RG method. In Sec. III A, using our method, the high concentration second virial coefficient is calculated. Further properties such as radius of gyration, screening length, and structure factor for finite concentration solution are obtained.

This paper is organized as follows. In order to illustrate the membrane scaling theory with finite concentration in solution, we start from the scaling relations for a single membrane. The scaling relations for a single tethered surface are described in Sec. II. The scaling relations for a finite concentration solution are presented in Sec. III. Our results are concluded in Sec. IV.

II. SCALING RELATIONS FOR SINGLE MEMBRANE

Let us consider the simplest case of a single membrane in an infinite dilute solution. The interesting properties of membrane solution are sensitive to its long range properties. It is therefore adequate to represent the membrane with the familiar Gaussian membrane model [4,5]. The theoretical study of a self-avoiding (SA) polymerized membrane is centered around a model of tethered self-avoiding manifolds directly inspired by the Edward model for polymers. In the generalized Edward model, the 2D surfaces are generalized to an intrinsically D-dimensional manifold, i.e., D-dimensional connected networks. The nodes of this D-dimensional connected network are labeled by internal continuous coordinates $x\in\mathbb{R}^D$ and are embedded in an external d-dimensional space with position vector $\vec{r}(x)\in\mathbb{R}^d$. The partition function is then obtained by summing over all configurations of the surface $\vec{r}(x)$, to which a term accounting for the excluded volume interaction has been added,

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$$Z=\prod_{i=0}^{N}\int D\vec{r}_i\exp\left[-\frac{1}{2}K\int_0^{L^D}d^Dx\sum_{a=1}^{D}\left(\frac{\partial\vec{r}}{\partial x_a}\right)^2-\frac{1}{n!}\nu_n\int\prod_{i=1}^{n}d^Dx_i\prod_{j=1}^{n-1}\delta^d[\vec{r}(x_j)-\vec{r}(x_n)]\right]$$
$$\times\left\{\prod_{i=0}^{N}\int D\vec{r}_i\exp\left[-\frac{1}{2}K\int_0^{L^D}d^Dx\sum_{a=1}^{D}\left(\frac{\partial\vec{r}}{\partial x_a}\right)^2\right]\right\}^{-1},\tag{1}$$

with an elastic Gaussian term and a self-avoiding n-body δ potential with excluded volume parameter $\nu_n>0$. The internal D-dimensional ‘‘volume’’ V is represented by $|x^D|$.

Let us perform the rescalings $r'(x')=a_0r(x)$ and $x'=b_0x$. The partition function, Eq. (1), can then be rewritten as

$$Z=\prod_{i=0}^{N}\int D\vec{r}_i'\exp\left[-\frac{1}{2}K\frac{b_0^{2-D}}{a_0^2}\int_0^{(b_0L)^D}d^Dx'\sum_{a=1}^{D}\left(\frac{\partial\vec{r}'}{\partial x_a'}\right)^2-\frac{1}{n!}\frac{a_0^{(n-1)d}}{b_0^{nD}}\nu_n\int_0^{(b_0L)^{nD}}\prod_{i=1}^{n}d^Dx_i'\prod_{j=1}^{n-1}\delta^d[\vec{r}(x_j')-\vec{r}(x_n')]\right]$$
$$\times\left\{\prod_{i=0}^{N}\int D\vec{r}_i'\exp\left[-\frac{1}{2}K\frac{b_0^{2-D}}{a_0^2}\int_0^{(b_0L)^D}d^Dx'\sum_{a=1}^{D}\left(\frac{\partial\vec{r}'}{\partial x_a'}\right)^2\right]\right\}^{-1}.\tag{2}$$

As we propose the following dimensionless quantities $b_0=b/l$, $a_0=a/l$, $N=L/l$, $a=b^{(2-D)/2}$, Eq. (2) corresponds to a membrane with $L'=bN$, $K'=1$, and $\nu_n'=b^{(n-1)(2-D)d/2-nD}l^{(n-2)D-(n-1)d}\nu_n$. Here n means n-body δ interaction. N is the polymerization index in each one dimension. The Kuhn length l is defined by $l=K^{-1/D}$. This partition function can then be displayed as a function of two dimensionless variables Nb and $b^{(n-1)(2-D)d/2-nD}l^{(n-2)D-(n-1)d}\nu_n$ such that

$$Z=Z(bN,\nu_nb^{(n-1)(2-D)d/2-nD}l^{(n-2)D-(n-1)d}).\tag{3}$$

A similar transformation can be applied to the radius of gyration of a membrane. In order to calculate the radius of gyration of a membrane, one needs the averaged square distance [10]

$$C(x_1,x_2)=\langle[\vec{r}(x_1)-\vec{r}(x_2)]^2\rangle$$
$$=\int D\vec{r}[\vec{r}(x_1)-\vec{r}(x_2)]^2P\Bigg/\left(\int D\vec{r}\,P\right),\tag{4}$$

such that the generalized radius of gyration [10] can be defined, in analogy to polymers, by the integration of the averaged square distance

$$R_G^2=\frac{1}{L^{2D}}\int d^Dx_1\,d^Dx_2C(x_1,x_2),\tag{5}$$

where the probability distribution is given by

$$P=\exp\left\{-\frac{1}{2}K\int_0^{L^D}d^Dx\left[\sum_{a=1}^{D}\left(\frac{\partial\vec{r}}{\partial x_a}\right)^2\right]\right.$$
$$\left.-\frac{1}{n!}\nu_n\int\prod_{i=1}^{n}d^Dx_i\prod_{j=1}^{n-1}\delta^d(\vec{r}(x_j)-\vec{r}(x_n))\right\}.\tag{6}$$

Since the numerator and denominator in Eq. (4) have the same factor $D\vec{r}$, the generalized radius of gyration satisfies the scaling relation

$$R_G^2=l^2N^{-2D}b^{-2-D}f(Nb,\nu_nl^{(n-2)D-(n-1)d}$$
$$\times b^{(n-1)(2-D)d/2-nD}),\tag{7}$$

where f is an unknown function. For $Nb=1$, Eq. (7) is specialized to

$$R_G^2=l^2N^{2-D}f(\nu_nl^{(n-2)D-(n-1)d}N^{nD-(n-1)(2-D)d/2}),\tag{8}$$

which is the fundamental scaling relation for membrane dimension at infinite dilute solution. From Eq. (8) [5], we define the dimensionless interaction parameter z as

$$\nu_nl^{(n-2)D-(n-1)d}N^{[(2-D)/2](n-1)\{[n/(n-1)][2D/(2-D)]-d\}}.$$

Without the N dependence, $z\sim N^0$ leads to a multicritical line $d^*(n,D)$ in the (d,D) plane, i.e.,

$$d^*(n,D)=\frac{n}{n-1}\frac{2D}{2-D}=\frac{n}{n-1}D_H,$$

where D_H is the Gaussian Hausdorff dimension. With this critical line d^*, the ε value in the RG theory is defined as $\varepsilon=2nD-(n-1)(2-D)d$. With this ε value and the (d,D) plane, we can investigate the radius of gyration. When the self-avoidance is negligible, i.e., $z\rightarrow0$, the radius of gyration approaches $R_G^2=l^DL^{2-D}f(1,0)\sim L^{2\upsilon}$ where $\upsilon=(2-D)/2$. However, in the self-avoidance regime, $z\gg1$, the corresponding power law is $R_G^2=l^DL^{2-D}z^x\sim L^{2\upsilon}$ where $\upsilon=(2-D)/2+O(\varepsilon)$ [7]. With $z\sim L^{nD-(n-1)(2-D)d/2}$, we obtain the power $x=[1/nD-(n-1)(2-D)d/2]O(\varepsilon)$. Now we can consider the special case of two-body interaction, $n=2$, and the radius of gyration can be simplified to $l^2b^{-1}(Nb)^{-2}f(Nb,\nu_nb^{(d-4)/2}l^{-d})$ for $D=1$. For an infinite dilute solution, $b=N^{-1}$, i.e., $b=(L/l)^{-1}$, R_G^2 can be reduced to $l^DL^{2-D}f(\nu_nl^{-d}N^{2D-d(2-D)/2})$. From this argument, we find that the dimensionless interaction parameter z can be expressed in terms of the critical dimension $d^*=4D/(2-D)$ and is denoted by $z=N^{2D-d(2-D)/2}l^{-d}\nu_n=N^{(d^*-d)(2-D)/2}l^{-d}\nu_n$. When $d>d^*$, the effective interaction parameter is small, i.e., $z\rightarrow0$ for $N\rightarrow\infty$. Then the radius of gyration R_G^2 scales as $l^DL^{2-D}f(1,0)\sim L^{2\upsilon}$, where $\upsilon=(2$

$-D)/2$ and $f(1,0)\sim 1$; while for $d<d^*$, $z\rightarrow 0$ when the polymerization index $N\rightarrow 0$, i.e., short chain or small membrane, yields $f(1,z)\approx 1+O(z)$ and the radius of gyration R_G^2 scales as $l^D L^{2-D}[1+O(z)]$. In the regime $d<d^*$, the effective interaction parameter diverges, i.e., $z\rightarrow\infty$ for $N\rightarrow\infty$, and the self-avoidance is fully relevant. Recall that R_G^2 displays a power law dependence on N for $N\rightarrow\infty$. This implies that $R_G^2=l^2N^{2-D}\{z^x+O(z^{x'})\}$ for $x>x'$ and $z\gg 1$. The corresponding power of x is obtained by the definition $R_G^2=l^D L^{2-D}z^x\sim L^{2v}$, where $v=(2-D)/2+O(\varepsilon)$. With $z\sim L^{(2-D)/2}(d^*-d)$, we have $x=[4/(2-D)(d^*-d)]O(\varepsilon)$, where $O(\varepsilon)$ is obtained by Kardar and Nelson [7] as

$$O(\varepsilon)=\frac{\varepsilon}{2d^*+2^{3-d^*/2}\pi^{1/2}\Gamma(2+d^*/4)/\Gamma(\frac{1}{2}+d^*/4)}.$$

In the case of a three-body interaction, a similar result for $O(\varepsilon)$ is obtained in Ref. [11] by using the multilocal operator product expansion RG method, i.e., $O(\varepsilon)=a/2(ad_c+b)+O(\varepsilon^2)$, where $d_c=3/(2-D)$. a and b are the expansion coefficients of the renormalization constant of $\vec{r}(x)$ and the coupling constant, respectively.

III. SCALING RELATIONS AT FINITE CONCENTRATION

In this section we show the basic scaling relation for the osmotic pressure. The RG scaling results of a low concentration membrane solution are also obtained by Duplantier [5].

Let us consider the case of p membranes of dimension $L^D\times L^D$ in a volume V. The partition function for the p membrane is given by

$$Z_p=V^p\left(\prod_{\alpha=1}^{p}\prod_{i=0}^{N}\int_V D\vec{r}_{\alpha i}\right)\exp\left[-\frac{1}{2}K\sum_{\alpha=1}^{p}\int_0^{L^D}d^Dx\sum_{a=1}^{D}\left(\frac{\partial\vec{r}_\alpha(x)}{\partial x_a}\right)^2-\frac{1}{n!}\nu_n\sum_{\alpha,\alpha'=1}^{p}\int\prod_{i=1}^{n}d^Dx_i\prod_{j=1}^{n-1}\delta^d[r_\alpha(x_j)-r_{\alpha'}(x_n)]\right]$$
$$\times\left\{\left(\prod_{\alpha=1}^{p}\prod_{i=0}^{N}\int_V D\vec{r}_{\alpha i}\right)\exp\left[-\frac{1}{2}K\sum_{\alpha=1}^{p}\int_0^{L^D}d^Dx\sum_{a=1}^{D}\left(\frac{\partial\vec{r}_\alpha(x)}{\partial x_a}\right)^2\right]\right\}^{-1}. \tag{9}$$

The volume is rescaled as $b^{d(2-D)/2}\ l^{-d}$ and the variable change in Eq. (2) admits the following scaling relation:

$$Z_p=(b^{(D-2)/2}l)^{pd}Z_p(Nb,\nu_n b^{d(n-1)(2-D)/2-nD}l^{(n-2)D-(n-1)d},Vl^{-d}b^{d(2-D)/2}). \tag{10}$$

The grand partition function is defined via Z_p as

$$\Xi=\sum_{p=0}^{\infty}\frac{Z_p\lambda^p}{p!}=\sum_{p=0}^{\infty}\frac{(\lambda b^{-d(2-D)/2}l^d)^p}{p!}Z_p(Nb,\nu_n b^{d(n-1)(2-D)/2-nD}l^{(n-2)D-d(n-1)},Vl^{-d}b^{d(2-D)/2})$$
$$=\Xi(\lambda b^{-d(2-D)/2}l^d,Nb,\nu_n b^{d(n-1)(2-D)/2-nD}l^{(n-2)D-d(n-1)},Vl^{-d}b^{d(2-D)/2}), \tag{11}$$

where the fugacity $\lambda=e^{\beta\mu}$ and μ is the chemical potential. The averaged number of membranes, $\langle p\rangle$, in the volume V is generated by the grand partition function

$$\langle p\rangle=\lambda\frac{\partial\ln\Xi}{\partial\lambda}=f_1(\lambda b^{-d(2-D)/2}l^d,Nb,\nu_n b^{d(n-1)(2-D)/2-nD}l^{(n-2)D-d(n-1)},Vl^{-d}b^{d(2-D)/2}). \tag{12}$$

We may invert the last equation into

$$\lambda b^{-d(2-D)/2}l^d=f_2(\langle p\rangle,Nb,\nu_n b^{d(n-1)(2-D)/2-nD}l^{(n-2)D-d(n-1)},Vl^{-d}b^{d(2-D)/2}). \tag{13}$$

The osmotic pressure is defined via the grand partition function and obeys the scaling relation

$$\frac{\pi V}{k_BT}=f_3(\langle p\rangle,Nb,\nu_n b^{d(n-1)(2-D)/2-nD}l^{(n-2)D-d(n-1)},Vl^{-d}b^{d(2-D)/2}), \tag{14}$$

where f_3 is an unknown function with arguments. For an infinite dilute solution, the interaction between different membranes may be ignored; we obtain $\Xi=\Sigma_{p=0}^{\infty}1/p!(\lambda l^d b^{-d(2-D)/2})^p\ Z_1^p(Nb,b^{d(n-1)(2-D)/2-nD}l^{(n-2)D-d(n-1)}\nu_n)=\exp\{\lambda l^d b^{-d(2-D)/2}Z_1\}$, for $c\rightarrow 0$. With this grand partition function, the classical result is obtained, i.e., $\pi V/k_BT=\langle p\rangle$, in the low concentration limit. The corresponding f_3 is proportional to $\langle p\rangle$. Since $\langle p\rangle$ and $Vl^{-d}b^{d(2-D)/2}$ are extensive variables, we use the intensive ratio $\langle p\rangle/Vl^{-d}b^{d(2-D)/2}$ as the expansion variable in the osmotic pressure and obtain

$$\frac{\pi V}{k_BT}=\langle p\rangle f_\pi\left(\frac{\langle p\rangle}{Vl^{-d}b^{d(2-D)/2}},Nb,\nu_n l^{(n-2)D-d(n\text{-}1)}b^{d(n-1)(2-D)/2-nD}\right). \tag{15}$$

To express Eq. (15) in terms of concentration, we may define the membrane mass concentration as

$$c=\frac{M_A\langle p\rangle N}{N_A V}, \tag{16}$$

where M_A is the unit molecular weight, N_A is Avagadro's number, and V is the volume. Replacing $\langle p\rangle$ by c, one obtains

$$\frac{\pi}{k_BT}=\frac{cN_A}{M_AN}\,f_\pi\left(\frac{N_Al^d}{M_A}\,cb^{1-d(2-D)/2}\,\frac{1}{Nb},Nb,\nu_nl^{(n-2)D-d(n-1)}b^{d(n-1)(2-D)/2-nD}\right). \tag{17}$$

For $Nb=1$, Eq. (17) is recast into the basic scaling relation for the osmotic pressure:

$$\frac{\pi}{k_BT}=\frac{cN_A}{M_AN}\,f_\pi(N_Al^dcN^{d(2-D)/2-1}/M_A\,,\nu_nl^{(n-2)D-d(n-1)}N^{nD-d(n-1)(2-D)/2}). \tag{18}$$

A. Limits of scaling function f_π

In this section, we show the low and high concentration limits of the osmotic pressure.

The intensive variables in Eq. (18) can be redefined by the general parameters, i.e., $X=N_Al^dcN^{d(2-D)/2-1}/M_A$ and $Y=\nu_nl^{(n-2)D-d(n-1)}N^{nD-d(n-1)(2-D)/2}$, which govern the nonideality behavior of the solution thermodynamics $(N\rightarrow\infty)$. When the excluded volume effect is negligible, i.e., $Y\ll1$, the perturbation expansion in parameter Y is permissible and gives

$$\frac{\pi}{k_bT}=\frac{cN_A}{M_AN}\,\{f_{\pi,0}(X)+Yf_{\pi,2}(X)+O(Y^2)\}. \tag{19}$$

The random walk membrane model successfully describes membranes under ‘‘ideal’’ or theta conditions where the second virial coefficient is vanished. In the ‘‘theta’’ condition, i.e., $Y=0$, only the leading term of Eq. (19) survives. At low concentration, i.e., $X\ll1$, we may expand the osmotic pressure in terms of concentration $\pi/k_BT=(cN_A/M_AN)\{1+Xf_{\pi,1}(Y)+O(X^2)\}$. For a high concentration solution, $X\gg1$, $f_{\pi,0}$ is assumed to have a power law form that gives $\pi/k_BT=(cN_A/M_AN)X^{x_1}$. For sufficiently high concentration solution, $c\rightarrow\infty$, a given node of a membrane can no longer determine which nodes belong to that membrane or others. Hence, $\pi/k_BT=(cN_A/M_AN)X^{x_1}$ is N independent and the power is equal to $x_1=1/[d(2-D)/2-1]$. The corresponding scaling relation is proportional to $l^d\pi/k_BT=[(N_A/M_A)cl^d]d(2-D)/[d(2-D)-2]$ for $\nu_n\rightarrow0$ and $c\rightarrow\infty$. Note that this scaling relation is n independent. The critical concentration that is a demarcation between the low concentration and high concentration regimes is defined as $X=(N_A/M_A)c_0^*l^d\,N^{d(2-D)/2-1}\sim1$; i.e., the good solution demarcation is $(N_A/M_A)c^*l^d\sim N\,R_G^{-d}=N^{1-\upsilon d}$. The power law of osmotic pressure in the critical concentration regime is expressed by

$$\frac{\pi}{k_BT}=\frac{N_Ac}{M_AN}\,g_\pi(XY^y). \tag{20}$$

With the choice of y so that the argument XY^y involves c^*/c, i.e., the exponent for N is equal to $\upsilon d-1$, one obtains the single variable equation of state

$$\frac{\pi}{k_BT}=\frac{N_Ac}{M_AN}\,g_\pi\left(\frac{N_A}{M_A}\,cN^{d\upsilon-1}l^d[\nu_nl^{(n-2)D-d(n-1)}]^{[\upsilon D-d(2-D)/2]/[nD-d(n-1)(2-D)/2]}\right), \tag{21}$$

with $g_\pi(0)=1$. For dilute solution, i.e., $(N_Al^d/M_A)c\ll(N_Al^d/M_A)c^*$, Eq. (21) is expanded in terms of concentration

$$\frac{\pi}{k_BT}=\frac{N_Ac}{M_AN}+\left(\frac{N_Ac}{M_AN}\right)^2l^dN^{\upsilon d}[\nu_nl^{(n-2)D-(n-1)d}]^{[\upsilon D-d(2-D)/2]/[nD-d(n-1)(2-D)/2]}g'_\pi(0)+\cdots=\frac{N_Ac}{M_AN}+A_2(0)c^2+\cdots\ , \tag{22}$$

which provides the second virial coefficient as

$$A_2(0)=l^d\left(\frac{N_A}{M_A}\right)^2N^{\upsilon d-2}[\nu_nl^{(n-2)D-(n-1)d}]^{[\upsilon D-d(2-D)/2]/[nD-d(n-1)(2-D)/2]}g'_\pi(0)+\cdots\ . \tag{23}$$

To consider the special case of two-body interaction, $n=2$, we find that the second virial coefficient is equal to $l^d(N_A/M_A)^2N^{\upsilon d-2}[\nu_2l^{-d}]^{(2\upsilon-d)/(4-d)}g'_\pi(0)$ for polymer $D=1$ and $l^d(N_A/M_A)^2N^{\upsilon d-2}[\nu_2l^{-d}]^{\upsilon/2}g'_\pi(0)$ for membrane $D=2$. At low concentration, the osmotic pressure can be calculated by using the RG method [5]. Our method makes the extension to higher concentration available. In the regime $c\gg c^*$, the argument of g_π in the equation of state is large, the power law form is expressed as

$$\frac{\pi}{k_BT}\sim\frac{N_Ac}{M_AN}\left(\frac{N_Al^d}{M_A}\,cN^{\upsilon d-1}(\nu_nl^{(n-2)D-(n-1)d[\upsilon D-D(2-D)/2]/[nD-d(n-1)(2-D)/2]})\right)^x.$$

For very high concentration membrane solution, a node on a single membrane cannot be distinguished between nodes belonging to distant parts of its membrane or others. The osmotic pressure π depends on c and ν_n only, not on L. Therefore, the exponent of N is equal to zero. We obtain the scaling relation

$$\frac{\pi}{k_BT}\sim\left(\frac{N_A}{M_A}\right)^{\upsilon d/(\upsilon d-1)} l^{d/(\upsilon d-1)}c^{\upsilon d/(\upsilon d-1)}(\nu_n l^{(n-2)D-(n-1)d})^{[\upsilon D-d(2-D)/2]/[nD-d(n-1)(2-D)/2]\ 1/(\upsilon d-1)}.$$

Finally, the osmotic pressure of two-body interaction, $n=2$, takes the form

$$\frac{\pi}{k_BT}\sim l^{d/(\upsilon d-1)}(N_A/M_A)^{\upsilon d/(\upsilon d-1)}c^{\upsilon d/(\upsilon d-1)}[\nu_n l^{-d}]^{(2\upsilon-d)/(4-d)\ 1/(\upsilon d-1)}$$

for polymer $D=1$ and

$$\frac{\pi}{k_BT}\sim l^{d/(\upsilon d-1)}\left(\frac{N_A}{M_A}\right)^{\upsilon d/(\upsilon d-1)} c^{\upsilon d/(\upsilon d-1)}[\nu_2 l^{-d}]^{\upsilon/2\ 1/(\upsilon d-1)}$$

for membrane $D=2$.

B. Membrane size at finite concentration

In this section, we investigate the radius of gyration of the single ‘‘labeled’’ membrane in solution of concentration c.

We define the probability distribution function $G_c(R,L|\nu_n,c)$ for the end-to-end vector R of an extra single membrane $(\alpha=0)$ in a solution of concentration c as

$$\begin{aligned}G_c(R;L|\nu_n,c)=V^{p+1}&\left(\prod_{\alpha=0}^{p}\prod_{i=0}^{N}\int_V D\vec{r}_{\alpha i}\right)\delta^d(\vec{r}_0(x_1)-\vec{r}_0(x_2)-R)\exp\left[-\frac{1}{2}K\sum_{\alpha=0}^{p}\int_0^{L^D}d^Dx\sum_{a=1}^{D}\left(\frac{\partial\vec{r}_\alpha}{\partial x_a}\right)^2\right.\\&-\frac{1}{n!}\nu_n\sum_{\alpha,\alpha'=0}^{p}\int\prod_{i=1}^{n}d^Dx_i\sum_{j=1}^{n-1}\delta^d(r_\alpha(x_j)-r_{\alpha'}(x_n))\Bigg\{\left(\prod_{\alpha=0}^{p}\prod_{i=0}^{N}\int_V D\vec{r}_{\alpha i}\right)\\&\times\exp\left[-\frac{1}{2}K\sum_{\alpha=0}^{p}\int_0^{L^D}d^Dx\sum_{a=1}^{D}\left(\frac{\partial\vec{r}_\alpha}{\partial x_a}\right)^2\right]\Bigg\}^{-1}.\end{aligned}\tag{24}$$

Change the variables of Eq. (24) and set $b=N^{-1}$, we can rewrite Eq. (24) as

$$G_c(R,L|\nu_n,c)=f(\langle p\rangle,\nu_n b^{d(n-1)(2-D)/2-nD}l^{(n-2)-(n-1)d},Nb,b^{(2-D)/2}l^{-1}R,Vb^{d(2-D)/2}l^{-d}),\tag{25}$$

where f is an unknown function with arguments. The generalized radius of gyration is obtained by following the procedures from Eq. (4) to Eq. (8) and yields

$$R_{G,c}^2=l^2N^{-2D}b^{-2-D}\tilde{f}(\langle p\rangle,Nb,\nu_n b^{d(n-1)(2-D)/2-nD}l^{(n-2)D-d(n-1)},Vl^{-d}b^{d(2-D)/2}).\tag{26}$$

For $Nb=1$, Eq. (26) is expanded with an intensive variable that is the ratio between extensive variables $\langle p\rangle$ and $Vl^{-d}b^{d(2-D)/2}$:

$$R_{G,c}^2=l^2N^{2-D}\tilde{f}\left(\frac{N_A}{M_A}l^dcN^{d(2-D)/2-1},\nu_nN^{nD-d(n-1)(2-D)/2}l^{(n-2)D-d(n-1)}\right).\tag{27}$$

With Eq. (27), we may consider the limiting cases. At low concentration $c\rightarrow0$ or $X\rightarrow0$, the expansion of $R_{G,c}^2$ in terms of X yields $R_{G,c}^2=R_G^2+l^2N^{2-D}f_1(0,Y)\times X+\cdots$, where the leading term is the single membrane radius of gyration. When we neglect the excluded volume effect, the expansion of $R_{G,c}^2$ in terms of Y generates the scaling relation $R_{G,c}^2=l^2N^{2-D}\{1+f_2(X,0)\times Y+\cdots\}$. Taking into account of the limit $Y\gg1$, i.e., large excluded volume effect, f has an overall power law dependence of Y^x. The scaling ansatz of the generalized radius of gyration is a product of a single membrane radius of gyration with a correction term, i.e.,

$$R_{G,c}^2=R_G^2g_{R^2}(XR_G^d)=l^2N^{2-D}f_{R^2}(Y)g_{R^2}(X[f_{R^2}(Y)]^{d/2})=l^2N^{2-D}Y^x\{g_{R^2}(0)-XY^y|g'_{R^2}(0)|+O(c^2)\}.$$

In a good solution, at low concentration, $R_{G,c}^2$ is expanded as

$$l^2N^{2\upsilon}(\nu_n l^{(n-2)D-d(n-1)})^{(2\upsilon+D-2)/[nD-d(n-1)(2-D)/2]}$$
$$\times\{g_{R^2}(0)-(N_A/M_A)l^dcN^{d\upsilon-1}(\nu_n l^{(n-2)D-d(n-1)})^{[d\upsilon-d(2-D)/2]/[nD-d(n-1)(2-D)/2]}|g'_{R^2}(0)|+O(c^2)\}.$$

When the membrane concentration increases considerably, nodes on a given membrane experience an environment that becomes progressively dominated by nodes on other membranes. Thus, the membranes must find themselves in a theta environment as $c\to\infty$. In this limit, the argument of g_{R^2} in $R^2_{G,c}$ becomes large, and it obeys the power law with exponent x'. Hence, as $c\to\infty$, the radius of gyration $R^2_{G,c}$ becomes

$$l^2N^{2\upsilon}(\nu_n l^{(n-2)D-d(n-1)})^{(2\upsilon+D-2)/[nD-d(n-1)(2-D)/2]}$$
$$\times[(N_A/M_A)cl^dN^{d\upsilon-1}]^{x'}(\nu_n l^{(n-2)D-d(n-1)})^{\{[d\upsilon-d(2-D)/2]/[nD-d(n-1)(2-D)/2]\}}.$$

Since the membrane has overall Gaussian dimensions, the exponent on N is equal to D. Consequently, this gives $x'=(D-2\upsilon)/(d\upsilon-1)$. When we consider the special case of two-body interaction, $n=2$, we find that the radius of gyration $R^2_{G,c}$ is equal to $l^2N[(N_A/M_A)cl^d]^{(1-2\upsilon/\upsilon d-1)}(\nu_2 l^{-d})^{(4\upsilon+2)/(4-d)+[(2\upsilon d-d)/(4-d)](1-2\upsilon)/(\upsilon d-1)}$ for polymer $D=1$ and to $l^2N^2[(N^A/M_A)cl^d]^{(2-2g\upsilon)/(\upsilon d-1)}(\nu_2 l^{-d})^{\upsilon/2+(\upsilon/2)d(2-2\upsilon)/(\upsilon d-1)}$ for membrane $D=2$.

C. Screening length

The intramolecular excluded volume interaction is screened in a more concentrated polymer solution where the concept of screening length ξ is introduced. This ideal may be generalized to a membrane solution. Since ξ is a length, it must be scaled in the same manner as all length scales, $\langle R^2\rangle_c$. This implies the general scaling relation

$$R^2_{G,c}l^{-2}=(\xi l^{-1})^2=N^{2-D}f\left(\frac{N_A}{M_A}\,cl^dN^{d(2-D)/2-1},\nu_n l^{(n-2)D-(n-1)d}N^{nD-d(n-1)(2-D)/2}\right). \tag{28}$$

For a bad solution, $\nu_n\to0$ and $c\to\infty$, Eq. (28) obeys the power law form $(\xi l^{-1})^2=N^{2-D}X^{x_a}$, which is independent of N. The corresponding power x_a is equal to $(2-D)/[d(2-D)/2-1]$ and the screening length ξ is proportional to $l[(N_A/M_A)cl^d]^{(2-D)/[d(2-D)-2]}$. On the other hand, when considering a good solution, $Y\gg1$, we find that the screening length $(\xi l^{-1})^2$ is equal to $N^{2-D}f_{\xi^2}(Y)\times g_{\xi^2}(Xf_{\xi^2}(Y)^{d/2})$ and is proportional to R^2_G. In this good solution, the membrane dimension varies as $N^{2\upsilon}$. We obtain the power that is $(2\upsilon+D-2)/[nD-d(n-1)(2-D)/2]$ and the general scaling relation becomes

$$(\xi l^{-1})^2=N^{2\upsilon}(\nu_n l^{(n-2)D-d(n-1)})^{(2\upsilon+D-2)/[nD-d(n-1)(2-D)/2]}g_{\xi^2}$$
$$\times\left(\frac{N_A}{M_A}\,cl^dN^{d\upsilon-1}(\nu_n l^{(n-2)D-d(n-1)})^{[d\upsilon-d(2-D)/2]/[nD-d(n-1)(2-D)/2]}\right).$$

Increasing the concentration, i.e., $Y\gg1$ and $c\to\infty$, we have an analogous result for

$$(\xi l^{-1})^2\cong N^{2\upsilon}(\nu_n l^{(n-2)D-d(n-1)})^{(2\upsilon+D-2)/[nD-d(n-1)(2-D)/2]}$$
$$\times\left[\frac{N_A}{M_A}\,cl^dN^{d\upsilon-1}(\nu_n l^{(n-2)D-d(n-1)})^{[d\upsilon-d(2-D)/2]/[nD-d(n-1)(2-D)/2]}\right]^{x_b}$$

in a good solution. This equation is independent of N for high concentration solution. Hence we have the power $x_b=-2\upsilon/(d\upsilon-1)$, which gives the scaling relation for screening length

$$\xi=l\left(\frac{N_A}{M_A}\,cl^d\right)^{-\upsilon/(d\upsilon-1)}$$
$$\times(\nu_n l^{(n-2)D-d(n-1)(1-\upsilon-D/2)/\{d\upsilon-1)[nD-d(n-1)(2-D)/2]\}}.$$

In the special case of two-body interaction, $n=2$, we have the screening length

$$\xi=l[(N_A/M_A)cl^d]^{-\upsilon/(\upsilon d-1)}(\nu_2 l^{-d})^{(1-2\upsilon)/[(\upsilon d-1)(4-d)]}$$

for polymer $D=1$ and $\xi=l[(N_A/M_A)cl^d]^{-\upsilon/(\upsilon d-1)}(\nu_2\, l^{-d})^{(-1/4)\upsilon/(\upsilon d-1)}$ for membrane $D=2$.

D. Scattering function-structure factor

Light scattering and coherent elastic neutron scattering studies are measured by the dynamic structure factor, which is defined as

$$I(q)=\left\langle\sum_{\alpha,\alpha'}\int_0^{L^D}d^Dx_1\int_0^{L^D}d^Dx_2\right.$$
$$\left.\times\exp\{iq[\vec{r}_\alpha(x_1)-\vec{r}_{\alpha'}(x_2)]\}\right\rangle, \tag{29}$$

where the angular brackets denote the grand canonical ensemble and q is the momentum. The choice of $b=N^{-1}$ provides the general scaling relation

$$I(q)=f_q\left(\frac{N_A}{M_A}\,l^d cN^{d(2-D)/2-1},lqN^{(2-D)/2},\right.$$
$$\left.\nu_n l^{(n-2)D-d(n-1)}N^{nD-d(n-1)(2-D)/2}\right). \qquad (30)$$

To study the limiting case, we adopt the intensive variables in Sec. III A. For $Y\ll 1$, f_q can be expanded in a Taylor series in terms of Y, leading to the perturbation expansion $I(q)=f_{q,0}(X,lqN^{(2-D)/2})+Yf_{q,1}(X,lqN^{(2-D)/2})+\cdots$. For $Y\rightarrow 0$ and $c>c^*$, $I(q)$ is equal to $g_{q,0}(lqN^{(2-D)/2}X^a)$ and is independent of N. So we have the concentration expansion of $I(q)$, i.e., $I(q)=g_{q,0}\{lq[(N_A/M_A)cl^d]^{2-D/2-d(2-D)}\}$ $=g_{q,0}(\xi q)$. Expanding $[g_{q,0}(\xi q)]^{-1}$ in powers of q^2, the final spectrum is $I(q)=k/[l+(\xi q)^2]$ where k is a constant, for $c>c^*$. For $Y\gg 1$, $I(q)$ reduces to

$$g_q\left(\frac{N_A}{M_A}\,cN^{d\upsilon-1}l^{d+[d\upsilon-d(2-D)/2][(n-2)D-d(n-1)]/[nD-d(n-1)(2-D)/2]}\nu_n^{[d\upsilon-d(2-D)/2]/[nD-d(n-1)(2-D)/2]},\right.$$
$$\left.qN^{\upsilon}l^{1+[\upsilon-(2-D)/2][(n-2)D-d(n-1)]/[nD-d(n-1)(2-D)/2]}\nu_n^{[\upsilon-(2-D)/2]/[nD-d(n-1)(2-D)/2]}\right).$$

For $c>c^*$, $I(q)$ is N independent, the structure dynamic factor is reduced to $I(q)=g_q^{>}[(q\xi)^{1-d\upsilon/\upsilon}]$.

IV. CONCLUSION

In conclusion, we have developed a general scaling relation of scaling theory of the self-avoidance crumpled membrane. The method of obtaining the scaling relation of the radius of gyration relies on the possibility of introducing the averaged square distance. Furthermore, we have shown how the scaling relation of the osmotic pressure, radius of gyration and screening length at finite concentration can be obtained. The above properties in low concentration can be calculated by using the RG method. Our method provides an easy way to investigate the thermodynamic scaling relationship. The high concentration power law form is obtainable rather than using the RG method, for example, osmotic pressure. Further application of this method to the charged polymerized membrane is under investigation.

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